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CALCULUS 101

DERIVATIVES &

INTEGRALS

EVERYONE'S GUIDE TO THE

FEYNMAN LECTURES ON PHYSICS

BY

ROBERT L. PICCIONI, PH.D.

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Edited by Joan Piccioni

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GUIDE TO THE COSMOS

Calculus 101

Here, we explore the most basic procedures of calculus: derivatives and integrals. This branch of mathematics is both beautiful and essential to scientific literacy.

Knowing how to differentiate and integrate is a survival skill for physicists. Calculus will not only help you pass exams and succeed in your careers, but it will also change your entire worldview. After learning calculus, you will appreciate the majesty of the universe more profoundly than ever before.

Many of you will have already read some of this material in the ***Feynman Simplified*** series of eBooks. I hope some redundancy will not be unwelcome. For others who have not (yet) read ***Feynman Simplified***, all of this may be new, exciting, and perhaps frightening.

Do not expect to fully appreciate calculus after just one reading. Changing your worldview will take some time. For me, and for others I know, understanding seemed to come in one blinding flash. I vividly recall the moment I finally grasped the meaning of: How fast is a balloon expanding when it is 10 cm wide?

Enjoy this.

Learning calculus is one of the great experiences of a scientific education.

Thank you for sharing this experience with me.

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Chapter 12
Derivatives

Derivatives describe rate of change.

Since everything in the universe changes, derivatives are the foundation upon which science strives to describe all natural phenomena.

To clarify “rate of change”, let’s consider an example familiar to almost everyone: riding in a car.

Speed

Imagine that you are driving through the scenic Southwest U.S., when a policeman stops you and says: “You were speeding, doing 60 miles per hour (mph) in a 55 mph zone.”

Your feeble defense is that you could not have been going 60 miles per hour because you were driving for only 10 minutes, not a whole hour, and have only traveled 10 miles. He replies that at the *rate* you were going you *would have* gone 60 miles in one hour. But that is impossible, you say, because the road ends in another 15 miles. He retorts that your speed was 1 mile per minute, which is the same as 60 mph. You ask if there is a law in Arizona that prohibits driving 1 mile in one minute?

You are not talking your way out of a citation, but it is interesting to ponder: what exactly does “a speed of 60 miles per hour” really mean?

Speed is distance traveled divided by travel time. It does seem more sensible to average a car’s speed over minutes rather than hours. To describe your speed while passing another car, one should average over an even shorter time interval, such as one second.

Averaging over one second might be good enough for a car, but what about a ball falling from a great height? It turns out, 10 seconds after being dropped, the ball’s speed is 98 meters per second (m/s), and one second later, its speed increases to 106 m/s. Since that is a substantial change, we should probably compute its speed over an interval even less than one second. (I switched to metric units, the units scientists use.)

The ultimate answer is to define speed in terms of the *infinitesimal distance* traveled during an *infinitesimal time* interval. This concept, developed independently by Isaac Newton and Gottfried Leibniz, is the basis of *differential calculus*, the first “new math.”

Limits

By convention, we use the symbol *ds* to represent an infinitesimal change in distance and *dt* for an infinitesimal change in time. Calculus provides a precise definition of speed that we denote with the letter v:

$$v = \lim_{dt \to 0} \frac{ds}{dt}$$

This equation reads: v equals the *limit* of the ratio (ds/dt) as *dt goes to zero*.

Well-behaved ratios come closer and closer to a final value of v as we compute the ratio for smaller and smaller time intervals dt. The value v is the *asymptotic limit* of the ratio (ds/dt).

Not all ratios are well-behaved. The ratio 1/x is not well-behaved as x approaches 0, because it gets ever-larger and is infinite at x=0. But the ratio (sinx)/x is well-behaved as x approaches 0, because the numerator and denominator both approach 0 *at the same rate*. Let's look at some values.

x	sin(x)/x
1.00	0.8415
0.50	0.9589
0.25	0.9896
0.10	0.9983
0.03	0.99985
0.01	0.99998

For x>0 (x greater than zero), sin(x)/x gets closer and closer to 1 as x approaches zero, hence its *limit* equals 1.

(As a side note: in almost all of physics, angles in equations are measured in radians, not in degrees.)

Let's consider another example of this concept of limits: what is the speed of a falling ball 8 seconds after its release?

We will employ the customary equation relating distance s and time t for a falling body near Earth's surface, ignoring air resistance. That equation is:

$$s(t) = g\, t^2 /2$$

Here, g is the acceleration of gravity (9.8 m/sec^2), and s(t) denotes the distance s at time t, emphasizing that s is a *function* of t. We compare the distance s at time t with the distance at the infinitesimally later time t+dt.

$$s(t+dt) = g\, (t+dt)^2 /2$$

The infinitesimal distance traveled, ds, is the change in distance.

$$ds = s(t+dt) - s(t)$$
$$ds = g\, [\, (t+dt)^2 - t^2\,]/2$$
$$ds = g\, [\, t^2 + 2t\, dt + dt^2 - t^2\,]/2$$
$$ds = g\, [\, 2t\, dt + dt^2\,]/2$$
$$ds/dt = g\, [2t + dt]/2$$

Now, we take advantage of dt being extremely small, dt << 1 ("<<" means much less than). We can discard the dt term in the [ ]'s in the last equation because dt << 2t. Now, we can compute the speed as a function of time, v(t), and evaluate it at t = 8 sec:

$v(t) = ds/dt = g\,[2t]/2$
$v(t) = g\,t$
$v(8\ \text{sec}) = [9.8\ \text{m/sec}^2]\,[8\ \text{sec}] = 78\ \text{m/sec}$

Differentiation

The procedure performed above on s(t) is called *differentiation* or more specifically *taking the derivative of* s *with respect to* t. Differentiation is important enough to merit an entire symbology: dq is not d times q, but is rather a single symbol denoting a *differential*, a tiny increment of the variable q or a tiny range of values of q.

The d-symbology applies to any variable q, but coordinate differentials — such as dt, dx, dy, dz — are the most common. The expression ds/dt is the ratio of two differentials, and is called the *derivative* of s with respect to t.

The two d's in ds/dt **do not** cancel one another to leave s/t; ds/dt and s/t are entirely different expressions.

Any normal function or equation of physics can be differentiated. To differentiate X with respect to z, compute

$$dX/dz = [\,X(z+dz) - X(z)\,]\,/\,dz$$

and allow dz to become infinitesimal (take the limit as dz goes to zero).

Derivatives as Slopes

Derivatives have graphic significance. If we plot a function, such as y(x) = sinx shown in Figure 6-4, the derivative at each value of x is the slope of sinx at that x.

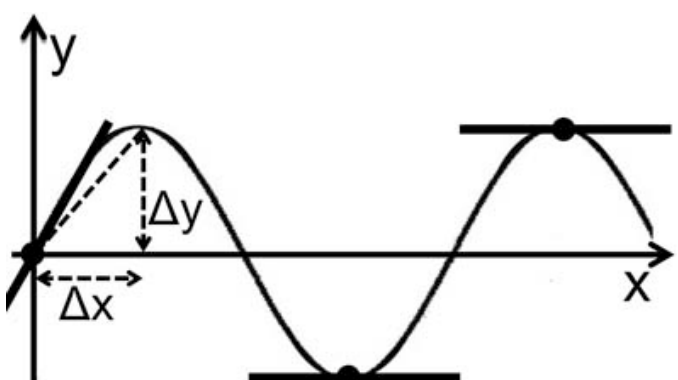

Figure 1-1 y(x) = sin(x)

At two of the points indicated by black dots, the derivative, dsin(x)/dx, equals 0; here the slope of sinx is also zero as shown by the horizontal tangent lines.

At x=0, Δy and Δx are indicated by dashed lines. We see that for a substantial value of Δx, the ratio

$\Delta y/\Delta x$, the slanted dashed line, differs from the tangent line at x=0. We show below that $\Delta y/\Delta x$ comes closer and closer to the true derivative as Δx gets closer and closer to zero.

The sine function can be expressed as an infinite series of which the first three terms are:

$\sin(x) = x - x^3/3! + x^5/5! - \ldots.$

where n! = n factorial = $1 \times 2 \times 3 \times \ldots \times n$

We can rewrite that as:

$\sin(x) = x \{1 - x^2/6 + x^4/120 - \ldots. \}$

The derivative at x=0 is the limit as Δx goes to zero of:

$d\sin(0)/dx = \{ \sin(\Delta x) - \sin(0) \} / \Delta x$
$d\sin(0)/dx = \sin(\Delta x) / \Delta x$
$d\sin(0)/dx = \Delta x\{1 - \Delta x^2/6 + \Delta x^4/120 - \ldots.\}/\Delta x$
$d\sin(0)/dx = 1 - \Delta x^2/6 + \Delta x^4/120 - \ldots.$

We see that for moderate values of Δx, the right side of the last equation is less than 1, but that as Δx goes to zero, Δx^2 and higher order terms become negligible, and the expression approaches 1, the true derivative.

The maxima and minima of any function are always at points at which its derivative is zero.

Now, we will step this up a notch, and consider acceleration a, the derivative of speed with respect to time. Continuing from above:

$v(t) = g\, t$
$v(t+dt) = g\, [t+dt]$
$dv = v(t+dt) - v(t)$
$dv = g\, [t+dt - t]$
$a = dv/dt = g = 9.8\text{m/sec}^2$

Since acceleration is the derivative of speed, and speed is the derivative of distance, we say acceleration is the *second derivative* of distance. Here, “second” means we differentiate twice. Note the symbology denoting the second derivative of s with respect to t:

$a = dv/dt = d^2s/dt^2$

First derivatives, like ds/dt, are ubiquitous in physics, and we often omit the word “first.” Second derivatives, like d^2s/dt^2, are less common. Third and higher order derivatives are rare. The third derivative of distance, d^3s/dt^3, is called *jerk*.

If your car speeds up with constant acceleration, you will be pressed back in your seat but will not be unduly uncomfortable. But if the driver alternately floors and releases the gas pedal, the resulting acceleration *changes* will toss you back and forth, which is very uncomfortable. That discomfort is due to jerk. Smooth rides are all about minimizing jerk.

General Rules of Differentiation

Differentiation is a *linear* operation that obeys these rules:

For any constant a and variable q:

$da/dq = 0$

For any functions F & G, constants a & b, and variable q:

$d(aF+bG)/dq = a\ dF/dq + b\ dG/dq$

$d(FG)/dq = G\ dF/dq + F\ dG/dq$

$d(FG)/dq = d(GF)/dq$

$d(F/G)/dq = (1/G)dF/dq - (F/G^2)dG/dq$

$d(F^n)/dq = n\ F^{n-1}\ dF/dq$

Derivatives of Common Functions

Here, we list the derivatives of the most common functions of physics: polynomials, trig functions, and exponentials. In the following sections, we show the derivations of each derivative. Learning requires practice, so if this material is new to you, study the derivations carefully and then try some on your own.

x^n : $dx^n/dt = n\ x^{n-1}\ dx/dt$

sin(x) : $d \sin(x) / dt = + \cos(x)\ dx/dt$

cos(x) : $d \cos(x) / dt = - \sin(x)\ dx/dt$

e^x : $d\ e^x / dt = e^x\ dx/dt$

ln(x) : $d \ln(x) / dt = (1/x)\ dx/dt$

Proof of General Rules

We prove here the general rules for derivatives that are summarized above.

For brevity, I will write "P=>Q" to denote "in the limit that P goes to Q".

The definition of the derivative of F with respect to q is:

as dq=>0: $dF/dq = \{ F(q+dq) - F(q) \} /dq$

Hence,

$F(q+dq) = F(q) + dq\ dF/dq$

Also for brevity, F and G will mean F(q) and G(q).

d(aF+bG)/dq

= { aF(q+dq) – aF + bG(q+dq) – bG}/dq

= { aF + dq a dF/dq – aF
+ bG + dq b dG/dq – bG }/dq

= a dF/dq + b dG/dq

d(FG)/dq = { F(q+dq)G(q+dq) – FG}/dq

= { [F + dq dF/dq] [G + dq dG/dq] – FG}/dq

= { FG + dq G dF/dq + dq dG/dq
+ dq^2 dF/dq dG/dq– FG}/dq

As dq=>0, we can drop terms of order dq^2, yielding:

= { dq G dF/dq + dq F dG/dq }/dq

= G dF/dq + F dG/dq

d(GF)/dq = { G(q+dq)F(q+dq) – GF}/dq

d(GF)/dq = { F(q+dq)G(q+dq) – FG}/dq

d(GF)/dq = d(FG)/dq

d(F/G)/dq = { F(q+dq) / G(q+dq) – F/G}/dq

= { [F + dq dF/dq] / [G + dq dG/dq] – F/G}/dq

= { [F + dq dF/dq]G – F[G + dq dG/dq] }
/ { [G + dq dG/dq] G dq}

As dq=>0, the denominator goes to G^2 dq.

= { FG + dq G dF/dq – FG – dq F dG/dq }
/ { G^2 dq}

= (1/G) dF/dq – (F/G^2) dG/dq

$d(F^n)/dq$ = $\{F^n(q+dq) – F^n\}/dq$

= $\{ (F + dq\, dF/dq)^n – F^n\}/dq$

Keeping only the lowest order terms in dq yields:

= $\{ (F^n + n\, dq\, F^{n–1}\, dF/dq) – F^n\}/dq$

= $n\, F^{n–1}\, dF/dq$

Derivative of x^n

dx^n/dx = limit $dx=>0$ $\{ [(x+dx)^n – x^n] / dx\}$

First, let's expand $(x+dx)^n$.

$(x+dx)^n = (x+dx) \times (x+dx) \times \dots (x+dx)$

Here, there are n terms in the product on the right. This is called a binomial expansion. In evaluating the right hand side, one chooses one of the two terms (x or dx) in each parentheses, and multiplies all those together. That product constitutes one of the 2^n terms that must be summed to include every combination of one term from each of the n parentheses.

Included in the 2^n terms are products with dx raised to the k power, for k from 0 to n. The number of product terms containing k factors of dx is the same as the number of combinations for picking k items from a list of n items, which is:

$n! / k!\,(n–k)!$

This means we can rewrite the right hand side of the prior equation as:

$(x+dx)^n = \Sigma_k \{x^{n–k}\, dx^k\, n! / k!\,(n–k)! \}$

Here, Σ_k represents the sum over all values of k from k=0 to k=n. Dropping all terms of order dx^2 and higher reduces the sum to:

$(x+dx)^n = x^n + nx^{n–1}dx$ + smaller terms

We now put that into the derivative equation.

dx^n/dx = limit $dx => 0$ $\{ [(x+dx)^n – x^n] / dx\}$

$= \{ [x^n + nx^{n–1}dx – x^n] /dx\}$

$dx^n/dx = nx^{n–1}$

$dx^n/dx\, (dx/dt) = nx^{n–1}\, (dx/dt)$

$dx^n/dt = nx^{n–1}\, dx/dt$

This result is valid even if n is not an integer, as we prove later.

Derivatives of Sine & Cosine

Recall the trig relations:

$\sin(A+B) = \sin(A) \cos(B) + \sin(B) \cos(A)$
$\cos(A+B) = \cos(A) \cos(B) – \sin(B) \sin(A)$

$d \sin(x) / dx = \lim \{\sin(x+dx) – \sin(x) \} / dx$

$\sin(x+dx) = \sin(x) \cos(dx) + \sin(dx) \cos(x)$

As dx=>0, cos(dx)=>1 and sin(dx)=>dx. Hence, as dx=>0:

$\sin(x+dx)–\sin(x) \Rightarrow \sin(x)+dx \cos(x) – \sin(x)$
$\sin(x+dx)–\sin(x) \Rightarrow dx \cos(x)$

$d \sin(x) / dx = \cos(x)$

Now for the cosine:

$d \cos(x) / dx = \lim \{\cos(x+dx) – \cos(x) \} / dx$

$\cos(x+dx) = \cos(x) \cos(dx) – \sin(dx) \sin(x)$

As dx=>0, cos(dx)=>1 and sin(dx)=>dx. Hence:

$\cos(x+dx) \Rightarrow \cos(x) – dx \sin(x)$
$\cos(x+dx) – \cos(x) \Rightarrow – dx \sin(x)$

$d \cos(x) / dx = – \sin(x)$

Derivative of e^x

The definition of e is:

$e = \text{limit } n\Rightarrow\infty \text{ of } (1+1/n)^{\mathbf{n}}$

$e^{\mathbf{x}} = \exp(x) = \lim n\Rightarrow\infty\ (1+1/n)^{\mathbf{xn}}$

Since exponents in physics can sometimes be quite elaborate, I often use exp{x} instead to $e^{\mathbf{x}}$ to improve eBook readability.

We now evaluate $(1+1/n)^{\mathbf{xn}}$ using the binomial expansion. The terms with the smallest powers of the infinitesimal quantity 1/n are:

$\exp(x) = 1 + (xn)(1/n) + (xn)(xn–1)/2n^{\mathbf{2}}$
$+ (xn)(xn–1)(xn–2)/3!n^{\mathbf{3}} + \ldots$

As xn=>∞, this becomes:

$\exp(x) = 1 + x + x^{\mathbf{2}}/2 + x^{\mathbf{3}}/3! + x^{\mathbf{4}}/4! + \ldots$

$d\ exp(x) / dx = 0 + 1 + 2x/2 + 3x^2/3! + 4x^3/4! + \ldots$

$d\ exp(x) / dx = 1 + x + x^2/2 + x^3/3! + \ldots$

$d\ exp(x) / dx = exp(x)$

Derivative of Natural Logarithm

By definition of the natural logarithm ln:

$x = exp(ln[x])$

Taking derivative of both sides with respect to x yields:

$dx/dx = exp(ln[x])\ d(ln[x])/dx$

$1 = x\ d(ln[x])/dx$

$d(ln[x])/dx = 1 / x$

Derivative of x^a

For any x and constant a:

$d (x^a) /dx = d (exp\{ a\ ln[x] \}) /dx$

$= (exp\{ a\ ln[x] \})\ d(a\ ln[x])/dx$

$= (x^a)\ a/x = a\ x^{a-1}$

Chapter 13
Integrals

Just as addition is the inverse of subtraction, integration is the inverse of differentiation. If the derivative of X equals Y, then the integral of Y equals X — well almost. Since the derivative of any constant is zero, a more precise statement is:

If the derivative of (X + any constant) = Y,
then the integral of Y = X + any constant

Generally, the so-called arbitrary constant of integration is determined by initial conditions, as we shall see.

Integrals solve problems that are the reverse of the problems that derivatives solve.

For example, we know from above that the equation for the speed of a falling ball at time t is:

$v(t) = g\ t$, with $g = 9.8\ m/sec^2$

Let's now ask: how far has the ball dropped at time t? Since the ball's speed is continuously changing, we cannot solve this problem with geometry or algebra. The only way to solve it is with *integral calculus*. (Integral and differential are the two main branches of calculus.)

Here is how it works. We learned above that $ds(t)/dt = v(t)$. This means:

$ds(t) = v(t)\ dt$

This equation reads: (the infinitesimal distance traveled at time t) equals (the ball's speed at time t) × (the infinitesimal time interval during which this change occurs). So, what we need to do is add up all those infinitesimal ds's. That is what integration does.

To obtain a precise result, we again need to take the limit as dt goes to zero. We perform that sum by integration, using the symbol ∫, an enlarged S derived from the Latin word *summa*. The general equation for finding the distance s travels from a changing but known speed v(t) is:

$s(t) = \int ds(t) = \int v(t)\ dt$

For the case of a falling ball:

$s(t) = \int g\ t\ dt = g\ t^2 /2 + C$

The arbitrary integration constant C represents our arbitrary choice in defining the location of s=0. The equation says the ball will accelerate with the same time dependence from any initial height; in a specific situation, we set that initial height with C. Here, we choose s=0 at time t=0.

We know the value of the above integral because we found above that the derivative of t^2 equals 2t, thus the integral of t equals $t^2/2$. To find out how far the ball has moved between time A and time B, we use the above equation to compute s(B) – s(A), as follows:

$s(B) – s(A) = g [B^2 – A^2] /2$

For example, the distance fallen between t = 4 seconds to t = 8 seconds equals:

$4.9 \text{ m/sec}^2 [64 \text{ sec}^2 – 16 \text{ sec}^2] = 235 \text{ m}$

Integrals, like derivatives, have graphic significance. Figure 2-1 shows a plot of the speed of a falling ball versus time, with the dotted line representing the equation: $v(t) = g t$.

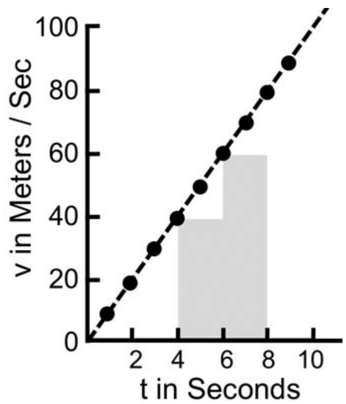

Figure 2-1 Plot of Velocity versus Time

The two shaded rectangles have heights of v(4sec) and v(6sec), and both have widths of 2 seconds. Since distance is speed × time, the area of each rectangle has units of meters/sec × sec = meters. The total area covered by both rectangles is:

Area = v(4sec)×2sec + v(6sec)×2sec

Area = (g×4sec)×2sec + (g×6sec)×2sec

Area = 9.8 m×(8+12) = 196 m

This total area of 196 m is a rough approximation to the distance the ball actually traveled between 4 and 8 seconds, which we calculated above to be 235m. The area is only approximate because the two rectangles do not cover the entire region under the line. The ball's velocity changes substantially during 2 seconds, leaving gaps above the rectangles. We could do better using shorter time intervals. With 4 rectangles each 1 second wide, the total area covered would be:

Area = [v(4)+v(5)+v(6)+v(7)]×1sec

Area = g×[4+5+6+7]×1

Area = 9.8 × 22 = 216 m

We are getting closer. Clearly the thing to do is to use an infinitesimal time interval dt, as shown in Figure 2-2.

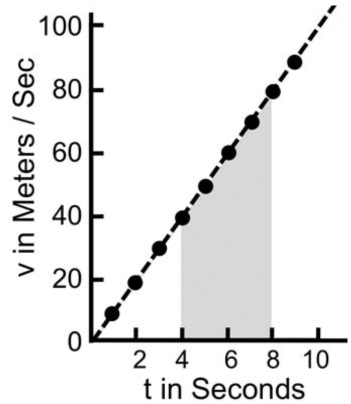

Figure 2-2 Shaded Area = Definite Integral

The shaded area under the line is the sum of an extremely large number of extremely thin rectangles, each of width Δt, which we can label n = 1, 2, 3, … We can now write:

Area = sum of $v(t_n) \times \Delta t$

Here, the sum is over all values of n

If we let n go to infinity, which means letting Δt go to zero, the sum of the areas of all the rectangles becomes the integral we previously calculated.

$s(t) = \int v(t)\, dt$

We see that integrals correspond to areas "under the curve" of a function, whereas derivatives correspond to the slope of that curve.

There are two types of integrals that are closely related. The above integral is called an *indefinite integral*: we integrate the function v(t) with respect to t, and get another function s(t), which like any normal function has a value at each value of t.

As you may have guessed, the other type of integral is called a *definite integral*. Here we select two values of t, A and B, and the definite integral yields the area under the curve between t=A and t=B. The definite integral is typically written:

$s(B) - s(A) = \int_A^B v(t)\, dt$

In the example in Figure 2-2, the definite integral from t = 4 sec to t = 8 sec is the shaded area under the curve, which we compute with the definite integral as follows:

$\int_A^B v(t)\, dt = s(B) - s(A)$
$= \{\int v(t)dt \text{ evaluated at } t=B\}$
$- \{\int v(t)dt \text{ evaluated at } t=A\}$

$\int_4^8 v(t)\, dt = s(8) - s(4)$
$= g\, \{t^2 \text{ evaluated at } t=8\} /2$
$- g\, \{t^2 \text{ evaluated at } t=4\} /2$

$= (4.9\text{ m/sec}^2)\, \{64\text{ sec}^2 - 16\text{ sec}^2\}$

$= 4.9$ m $\{48\} = 235$ m

Which equals what we calculated above for the distance traveled between t = 4 sec and t = 8 sec.

The result of every indefinite integral includes an arbitrary constant. The result of every definite integral has no arbitrary constant; the constant is the same at the limits A and B, and therefore cancels.

We described earlier the procedure to differentiate any expression. Unfortunately, there is no corresponding general procedure for integration. We learn how to do integrals with a haphazard reverse process: if we know that B is the derivative of A, then we know that A is the integral of B.

Mathematicians have differentiated a vast menagerie of functions and tabulated their results. If we want the integral of B, we search these tables hoping to find B listed as the derivative of some expression A. If B is listed, our answer is A. If not, we will be stuck calculating the areas of a vast number of rectangles, hopefully with a computer. Most physicists memorize many common integrals and keep extensive tables handy for others.

Here are some useful integrals that will solve most physics problems.

Powers

$\int x^n \, dx = x^{(n+1)} /(n+1)$

Here, n need not be an integer, but n cannot be –1.

$\int x^{-1} \, dx = \int (1/x) \, dx = \ln(x)$

Trig Functions (x in radians)

$\int \sin(x) \, dx = -\cos(x)$

$\int \cos(x) \, dx = +\sin(x)$

$\int \sin^2(x) \, dx = x/2 - \sin(2x) /4$

$\int \cos^2(x) \, dx = x/2 + \sin(2x) /4$

$\int \sin(x) \cos(x) \, dx = \sin^2(x) /2$

Exponentials

$\int \exp\{x\} \, dx = \exp\{x\}$

$\int \ln(x) = x \ln(x) - x$

Integration by Parts

For any two functions u and v:

$d(uv)/dx = u\, dv/dx + v\, du/dx$

$\int [d(uv)/dx]\, dx = \int [u\, dv/dx]\, dx + \int [v\, du/dx]\, dx$

$\int d(uv) = \int u\, dv + \int v\, du$

$uv - \int u\, dv = \int v\, du$